

GLOBAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL HORIZONTALLY FILTERED SIMPLIFIED BARDINA TURBULENCE MODEL ON A STRIP-LIKE REGION

ABSTRACT. We consider the 2D simplified Bardina turbulence model, with horizontal filtering, in an unbounded strip-like domain. We prove global existence and uniqueness of weak solutions in a suitable class of anisotropic weighted Sobolev spaces.

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1. INTRODUCTION

In the present paper we give some results mainly connected with the regularity and the long-time behavior of the viscous simplified Bardina turbulence model (with horizontal filtering) in a strip-like region $\Omega \subseteq \mathbb{R}^2$, aimed at proving existence and uniqueness of weak solutions in a suitable class of weighted Sobolev spaces.

The Bardina closure model for turbulence was introduced in 1980 by J. Bardina, J. H. Ferziger and W. C. Reynolds in [7], and later simplified and analyzed in [18] and in [12]. Indeed, the 3D simplified Bardina turbulence system was proposed in [18] as a regularization model, for small values of the scale parameter α , of the 3D Navier–Stokes equations for the purpose of numerical simulations. Analysis of the global behavior of the pertinent solutions in a bounded domain, with periodic boundary conditions, appears in [12]. Global well-posedness for the 2D simplified Bardina model was established in [13]. Again, in space-periodic domains, the inviscid simplified Bardina model is a regularizing system for the 3D Euler equations; this because it is globally well-posed and it approximates the 3D Euler equations without adding spurious regularizing terms (see [18]).

The behavior of the solutions for the simplified Bardina model (in both 2D and 3D cases) changes considerably depending on whether the integration domain is bounded. This is a basic point in studying general properties as regularity on the long-time period and dynamics (in particular, existence of attractors). More

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generally, this remark applies to the solutions of a broader class of dissipative systems (see, e.g., [4, 15, 21, 24]). In fact, unlike the case of bounded domains, for some types of solutions to PDEs in unbounded regions (such as spatially periodic patterns, travelling waves, etc...), we can not expect to have uniform control on the energy; rather the energy of these solutions may blow up to infinity. Again, due to the unboundedness of Ω , compactness for the semigroup solution operator can not be retrieved by using standard Sobolev embeddings (there are no compact inclusions). Hence, in this case, the standard choices for bounded domains of the phase space, as $L^p(\Omega)$, $W^{k,p}(\Omega)$ or $H^p(\Omega)$, $1 \leq p < +\infty$, $k \in \mathbb{N}$, do not seem appropriate.

Even in the promising situation in which the solutions are bounded as $|x| \rightarrow +\infty$ in Ω , i.e. they belong to $L^\infty(\Omega)$, the study of their behavior is not necessarily simplified since this space is analytically awkward to use: on one hand, strong requirements on the initial data are needed to have solutions in such a space; on the other hand, the study of dynamics in this phase space results to be more intricate since one does not have at disposal analytical semigroups, maximal regularity properties for semigroups, etc...

A reasonable alternative is using weighted Sobolev spaces (see, e.g., [1, 5, 21]) that, in principle, can contain sufficiently regular, spatially bounded solutions on the long-time period. In such a situation it is possible to study the semigroup generated by the considered system and to check whether it admits a global attractor in a suitable weighted phase space. A main advantage of this approach is that weighted Sobolev spaces are rather handy to use since they enjoy regularity, interpolation and embedding properties which are similar to those of the usual Sobolev spaces $W^{k,p}(\Omega)$ for bounded domain.

However, proving estimates in such spaces is more complicated than in the standard ones and, for our analysis, we find convenient to follow the same path as in [6] (see also [14, 22]). In so doing, we consider the 2D Navier–Stokes system in terms of a stream function, v , and derive formally the 2D simplified Bardina with horizontal filtering.

We now introduce the considered 2D simplified Bardina model for the potential v connected with the vector field $\mathbf{v} = (v_1, v_2)$ (here, \mathbf{v} is a regularizing vector field associated with the velocity field, \mathbf{u} , of the 2D Navier–Stokes equations (1.3) below, i.e. $\mathbf{v} \approx \mathbf{u}$ and $v_1 = \partial_2 v$, $v_2 = -\partial_1 v$), on the strip-like region $\Omega \subset \mathbb{R}^2$, i.e.:

$$(1.1) \quad \begin{aligned} (1 - \alpha^2 \partial_1^2) \Delta \partial_t v + B(v, v) - \nu (1 - \alpha^2 \partial_1^2) \Delta^2 v &= g, & x \in \Omega \text{ and } t \in \mathbb{R}^+, \\ v|_{t=0} &= v_0(x), & x = (x_1, x_2) \in \Omega, \\ v(x, t) &= 0, \nabla v(x, t) = 0, \partial_1 \nabla v(x, t) = 0, & x \in \partial\Omega \text{ and } t \in \mathbb{R}^+, \end{aligned}$$

where $B(v, v) := \partial_2 v \partial_1 \Delta v - \partial_1 v \partial_2 \Delta v$, $\alpha > 0$ is a scale parameter, $\nu > 0$ is the kinematic viscosity, g is a forcing term, and the domain Ω is defined by the following inequalities (see [6, 14, 22]):

$$(1.2) \quad b_1(x_1) \leq x_2 \leq b_2(x_1), \quad x_1 \in \mathbb{R},$$

where b_1 and b_2 are twice continuously differentiable functions bounded over the entire x_1 -axis according to

$$\begin{aligned} -M &\leq b_1(x_1) \leq x_2 \leq b_2(x_1) \leq M, \quad x_1 \in \mathbb{R}, \\ |b'_i(x_1) + b_i(x_1)| &\leq c, \quad i = 1, 2. \end{aligned}$$

In order to formally derive system (1.1), we consider the 2D Navier–Stokes equations in the space periodic setting $\Omega = \mathbb{T}^2$ (although it would be sufficient to consider periodicity only in the x_1 -direction), i.e.

$$(1.3) \quad \begin{aligned} \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla \pi &= \mathbf{f}(x, t), & x \in \Omega \text{ and } t \in \mathbb{R}^+, \\ \nabla \cdot \mathbf{u} &= 0, & x \in \Omega \text{ and } t \in \mathbb{R}^+, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, & x \in \Omega, \end{aligned}$$

where $\mathbf{u}(x, t) = (u_1, u_2)$ is the velocity field, $\pi(x, t)$ denotes the pressure, $\mathbf{f}(x, t) = (f_1, f_2)$ is the external force, and $\nu > 0$ the kinematic viscosity.

First, we rewrite the Navier–Stokes equations (1.3) in terms of the vorticity $\xi := \text{curl } \mathbf{u} := \partial_1 u_2 - \partial_2 u_1 \in \mathbb{R}$ and then we introduce the stream function ω associated to the velocity field \mathbf{u} , i.e. a scalar function $\omega \in \mathbb{R}$ such that $\mathbf{u} = \text{curl } \omega = (\partial_2 \omega, -\partial_1 \omega) \in \mathbb{R}^2$ (notice that $\xi = -\Delta \omega$), to get

$$(1.4) \quad \begin{aligned} \Delta \partial_t \omega + B(\omega, \omega) - \nu \Delta^2 \omega &= g, \\ \omega|_{t=0} &= \omega_0, \end{aligned}$$

where the bilinear operator B is as above (i.e. $B(\omega, \omega) = \partial_2 \omega \partial_1 \Delta \omega - \partial_1 \omega \partial_2 \Delta \omega$) and $g = \partial_2 f_1 - \partial_1 f_2$.

For a function w , we introduce the horizontal filter (related to the horizontal Helmholtz operator), given by

$$(1.5) \quad A_h = I - \alpha^2 \partial_1^2, \text{ and } \bar{w}^h := A_h^{-1} w.$$

As discussed in [3, 16, 17], from the point of view of the numerical simulations, this filter is less memory consuming with respect to the standard one. Further, another interesting feature of this filter is that, even in the case of domains which are not periodic in the vertical direction, there is no need to introduce artificial boundary conditions for the Helmholtz operator (see, e.g., [3, 8, 9, 10, 11]).

We set $v := \bar{w}^h$ (and $(v_1, v_2) = \mathbf{v} := \bar{\mathbf{u}}^h$, with $v_1 = \partial_2 v$, $v_2 = -\partial_1 v$) and solve the interior closure problem by using the approximations

$$\overline{B(\omega, \omega)}^h \approx \overline{B(\bar{w}^h, \bar{w}^h)}^h =: \overline{B(v, v)}^h,$$

to get the following initial value problem:

$$\begin{aligned} \Delta \partial_t v + \overline{B(v, v)}^h - \nu \Delta^2 v &= \bar{g}^h, \\ v|_{t=0} &= \bar{\omega}_0^h. \end{aligned}$$

By applying the operator $A_h = I - \alpha^2 \partial_1^2$ to the above system, term by term, and considering the obtained equations on the channel-like domain described by (1.2) (introducing suitable boundary conditions), we get (1.1). Here and in the sequel, for simplicity, we always assume that $g(x, t) = g(x)$.

Set the following anisotropic Sobolev spaces:

$$H_0^{2,h} = \{f \in W^{1,2}(\Omega) : \partial_1 \nabla f \in L^2(\Omega), \nabla \cdot f = 0 \text{ and } f|_{\partial\Omega} = 0\},$$

and

$$H^{3,h} = \{f \in W^{2,2}(\Omega) : \partial_1 \Delta f \in L^2(\Omega)\}.$$

As first step in our analysis we provide an existence theorem to (1.1) in standard anisotropic Sobolev spaces. In this case we deal with a proper class of weak solutions to the considered problem (see Definition 3.1 below). This result reads as follows.

Theorem 1.1. *Let $v_0 \in H^{3,h} \cap H_0^{2,h}$ and $g \in L^2$. Then, there exists a unique weak solution v of the problem (1.1).*

In the main theorem of the paper we prove that the global weak solution $v = v(t)$ in Theorem 1.1 is actually defined in a suitable class of anisotropic weighted Sobolev spaces (see Theorem 3.1 below).

In proving Theorem 3.1 we do not follow directly the scheme behind the standard Aubin–Lions lemma, rather we use a different compactness method (see [23, Corollary 2.34], see also Lemma 5.2 below) by which we perform our analysis on approximating open bounded subsets \mathcal{O} of Ω . This result allows us to surmount the difficulties due to the boundary conditions and the unboundedness of the considered domain Ω .

The results obtained here open the way to the analysis of dynamics in terms of attractors; this will be the matter of a forthcoming paper.

Plan of the paper. In Section 2 we introduce the main notation and we also give some preliminary results. In Section 3 we give the precise definition of weak solution, we state our main result (Theorem 3.1) and we also present some remarks on the existence of weak solutions. Section 4 is devoted to the proof of Theorem 1.1. In Section 5, we study problem (1.1) in suitable Sobolev weighted spaces proving Theorem 3.1. Finally, the appendix is dedicated to the properties of the weight functions used to define the weighted Sobolev spaces used in Theorem 3.1.

2. NOTATION AND PRELIMINARY RESULTS

In what follows, we denote by $L^p := L^p(\Omega)$, and $W^{k,p} := W^{k,p}(\Omega)$, with $H^k := W^{2,k}$, $k, p \in \mathbb{N}$, the usual Lebesgue and Sobolev spaces, respectively. Also, we denote by (\cdot, \cdot) and $\|\cdot\|$ the standard L^2 -inner product and norm in $L^2(\Omega)$, respectively. We denote by $(H^k)'$ the dual space to H^k , $k, p \in \mathbb{N}$, and this notation will be adapted in a straightforward manner, when it makes sense, to the further spaces that will be introduced in the sequel.

Given a Banach space X , for $p \in [1, \infty)$, we denote the usual Bochner spaces $L^p(0, T; X)$ with associated norm $\|f\|_{L^p(0, T; X)}^p := \int_0^T \|f(s)\|_X^p ds$ (the lower bound of $\|f(s)\|_X$ if $p = \infty$), with $\|\cdot\|_X$ the norm of X .

Hereafter, C will denote a dimensionless constant which might depend on the shape of the domain Ω and that may assume different values, even in the same line.

Let us introduce the following function spaces:

$$\begin{aligned} H &:= \{f \in L^2(\Omega) : \nabla \cdot f = 0 \text{ and } f = 0 \text{ on } \partial\Omega\}, \\ H^{1,h} &:= \{f \in L^2(\Omega) : \partial_1 f \in L^2(\Omega)\}, \\ H^{2,h} &:= \{f \in H^{1,h} : \partial_1 \nabla f \in L^2(\Omega)\}, \\ H^{3,h} &:= \{f \in H^{2,h} : \partial_1 \Delta f \in L^2(\Omega)\}, \end{aligned}$$

and $H_0^{l,h} := H^{l,h} \cap H$, $l = 1, 2, 3$.

2.1. Weighted Sobolev spaces: Basic properties and related inequalities.

Here, we consider a family of functions $\varphi_\rho(x, \alpha_1, \alpha_2, \epsilon, \gamma) = \varphi(x, \alpha_1, \alpha_2, \epsilon, \rho, \gamma)$ enjoying the following properties, analogue to those listed in [14, §2.2, (A), pg. 383]

(see also [6]):

$$\begin{aligned}
\varphi &\geq 1, \quad \varphi(x, \alpha_1, \alpha_2, \epsilon, \rho, \gamma) = \varphi(\epsilon x, \alpha_1, \alpha_2, 1, \rho, 1)^\gamma, \\
\varphi(x, \alpha_1, \alpha_2, 1, \rho, \gamma) &\text{ does not depend on } \rho \text{ if } |x| \leq \rho, \\
\varphi(x, \alpha_1, \alpha_2, 1, \rho, \gamma) &= \varphi(\rho + 1, \alpha_1, \alpha_2, 1, \rho, \gamma) \text{ as } |x| \geq \rho + 1 \\
\varphi(x, \alpha_1, \alpha_2, \epsilon, \rho_1, \gamma) &\geq \varphi(x, \alpha_1, \alpha_2, \epsilon, \rho_2, \gamma) \text{ for } \rho_1 \geq \rho_2 \geq 1, \gamma \geq 0, \\
\lim_{\rho \rightarrow +\infty} \varphi(x, \alpha_1, \alpha_2, 1, \rho, \gamma) &= (1 + |x_1|^{\alpha_1} + |x_2|^{\alpha_2})^{\frac{\gamma}{2}} =: \phi(x, \alpha_1, \alpha_2, 1, \gamma/2),
\end{aligned}$$

where we have introduced the anisotropic weight function $\phi(x, \alpha_1, \alpha_2, \epsilon, \gamma) = (1 + |\epsilon x_1|^{\alpha_1} + |\epsilon x_2|^{\alpha_2})^\gamma$ (see, e.g., [6, 15]), which is suitable for the anisotropic Sobolev spaces we will consider in the sequel (see the definition of $H_\gamma^{l,h}$, $l = 1, 2, 3$, below).

For the remainder of the paper we always assume $\alpha_1 = 3$, $\alpha_2 = 2$ and we use the compact notations $\phi := \phi(x, \epsilon, \gamma) := \phi(x, 3, 2, \epsilon, \gamma)$ and $\varphi := \varphi(x, \epsilon, \rho, \gamma) := \varphi(x, 3, 2, \epsilon, \rho, \gamma)$.

Again, arguing as in [6, 14], we take $\psi := \varphi^{1/2}$. Notice that we can choose φ so that

$$(2.1) \quad |\partial^\beta \psi^2| \leq C\epsilon^{|\beta|} \psi^2 \text{ for every multi-index } \beta = (\beta_1, \beta_2), \quad |\beta| \leq 3 \text{ and } \beta_2 \leq 2.$$

This property will play a crucial role in the subsequent computations.

We denote by $H_\gamma^l := H_\gamma^l(\Omega)$ the space of functions equipped with the following norm:

$$\|v\|_{l,\gamma}^2 := \sum_{|\beta| \leq l} \|\partial^\beta v\|_\gamma^2,$$

where

$$\|v\|_\gamma^2 := \int_\Omega |v|^2 (1 + |x_1|^3 + |x_2|^2)^\gamma dx, \quad \gamma > 0 \quad \text{and} \quad \partial^\beta := \frac{\partial^{\beta_1 + \beta_2}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}, \quad (\beta_1, \beta_2) \in \mathbb{N}^2.$$

Using the above notation, we introduce the further spaces

$$\begin{aligned}
H_\gamma &:= H_{0,\gamma} := \{f \in H : \|f\|_\gamma < +\infty\}, \\
H_\gamma^{1,h} &:= \{f \in H : \|\partial_1 f\|_\gamma < +\infty\}, \\
H_\gamma^{2,h} &:= \{f \in H^{1,h} \cap H : \|\partial_1 \nabla f\|_\gamma < +\infty\}, \\
H_\gamma^{3,h} &:= \{f \in H^{2,h} \cap H : \|\partial_1 \Delta f\|_\gamma < +\infty\}.
\end{aligned}
\tag{2.2}$$

Let us recall the following results taken from [6] (see also [14]).

Proposition 2.1. *If $v \in H_0^1(\Omega)$, then*

$$(2.3) \quad \|\psi \nabla v\| - \|\nabla(\psi v)\| \leq C\epsilon \|\psi v\|.$$

If $v \in H_0^2$, then it holds true that

$$(2.4) \quad \|\psi \Delta v\| - \|\Delta(\psi v)\| \leq C\epsilon \|\psi|v| + \psi|\nabla v|\|.$$

Next, we have a weighted version of the classical Poincaré inequality.

Proposition 2.2. *Let $v \in H_0^1$ with $v|_{\partial\Omega} = 0$. Then it holds true that*

$$(2.5) \quad \|\psi v\| \leq 2\lambda_1^{-1} \|\psi \nabla v\|.$$

Let ϵ in the definition of φ be sufficiently small. Let $v \in H^2 \cap H_0^1$. Then

$$(2.6) \quad \|\psi \nabla v\| \leq 2\lambda_1^{-\frac{1}{2}} \|\psi \Delta v\|.$$

Proposition 2.3. *It holds true that*

$$(2.7) \quad \|v\|_{1,\gamma} \leq \|\psi \nabla v\| \quad \forall v \in H_\gamma^1, \quad v|_{\partial\Omega} = 0,$$

and

$$(2.8) \quad \|v\|_{2,\gamma} \leq \|\psi \Delta v\| \quad \forall v \in H_\gamma^2, \quad v|_{\partial\Omega} = 0.$$

We also have the following controls in the L^4 -norm.

Proposition 2.4. *Let $v \in H_\gamma^1$, with $v|_{\partial\Omega} = 0$. Then*

$$(2.9) \quad \begin{aligned} \|\psi v\|_{L^4} &\leq C \|\nabla(\psi v)\| \leq C \|\psi \nabla v\| + C \|v \nabla \psi\| \\ &\leq C \|\psi \nabla v\| + C \epsilon \|v \psi\|. \end{aligned}$$

Further, if $v \in H_\gamma^2$, with $v|_{\partial\Omega} = 0$, then

$$(2.10) \quad \|\psi \nabla v\|_{L^4} \leq C \epsilon \|\psi \nabla v\| + C \|\psi \Delta v\|.$$

3. WEAK SOLUTIONS AND EXISTENCE RESULTS

Consider the simplified-Bardina model (1.1). Observe that the bilinear form

$$B(u, v) := \partial_2 v \partial_1 \Delta u - \partial_1 v \partial_2 \Delta u = \partial_1(\partial_2 v \Delta u) - \partial_2(\partial_1 v \Delta u)$$

is such that

$$\begin{aligned} (B(u, v), w) &= \int \partial_1(\partial_2 v \Delta u) w - \partial_2(\partial_1 v \Delta u) w = \int \partial_1 v \Delta u \partial_2 w - \partial_2 v \Delta u \partial_1 w \\ &= - \int \partial_1 w \Delta u \partial_2 v - \partial_2 w \Delta u \partial_1 v = -(B(u, w), v), \end{aligned}$$

and

$$(B(u, v), v) = 0,$$

where the second line is obtained integrating by parts and exploiting the boundary conditions. Here and in the sequel, unless stated otherwise, we drop the dx in the space-integrals to keep the notation as compact as possible.

We now give the following definition.

Definition 3.1. *Given $v_0 \in H^{3,h} \cap H_0^{2,h}$ and $g \in L^2(\Omega)$, we say that $v \in L_{\text{loc}}^\infty(\mathbb{R}; H_0^{2,h} \cap H^{3,h})$ is a weak solution of (1.1) if $v_t \in L_{\text{loc}}^2(\mathbb{R}; H_0^{2,h})$ and*

$$\begin{aligned} &(\nabla v_t, \nabla \hbar) + \alpha^2(\partial_1 \nabla v_t, \partial_1 \nabla \hbar) + \nu(\Delta v, \Delta \hbar) + \nu \alpha^2(\partial_1 \Delta v, \partial_1 \Delta \hbar) \\ &= (B(v, v), \hbar) - (g, \hbar) \end{aligned}$$

for every $\hbar \in H_0^{2,h} \cap H^{3,h}(\Omega)$, for a.e. $t \in \mathbb{R}$ (and the initial datum is assumed in weak sense).

In the next section we give a proof of Theorem 1.1 that guarantees existence and uniqueness of a weak solution to problem (1.1).

The anisotropic weighted Sobolev spaces introduced in (2.2) provide the appropriate functional framework for studying the existence of weak solutions to (1.1) enjoying extra regularity properties. Then, in Section 5 we prove our main result, that reads as follows.

Theorem 3.1. *Let $g \in H_{0,\gamma}$. Then, for any $v_0 \in H_\gamma^{3,h} \cap H_0^{2,h}$ and $T > \tau$ given, the weak solution v of (1.1) provided by Theorem 1.1 is such that $v \in L^\infty(\tau, T; H_\gamma^{2,h}) \cap L^2(\tau, T; H_\gamma^{3,h}) \cap C(\tau, T; H_\gamma^{1,h})$ and $v_t \in L^2(\tau, T; H_\gamma^{1,h})$.*

Corollary. *Under the hypotheses of Theorem 3.1 it holds true that $v \in L_{\text{loc}}^\infty(0, \infty; H_\gamma^{2,h}) \cap L_{\text{loc}}^2(0, \infty; H_\gamma^{3,h}) \cap C(0, \infty; H_\gamma^{1,h})$ and $v_t \in L_{\text{loc}}^2(0, \infty; H_\gamma^{1,h})$.*

4. EXISTENCE IN ANISOTROPIC SOBOLEV SPACES

This section is devoted to the proof of Theorem 1.1, which provides the existence of a unique weak solution of the problem (1.1). Since the proof follows standard methods, we proceed formally in order to find appropriate a priori estimates. A rigorous proof can be easily obtained by introducing a Galerkin approximation and finding similar estimates.

We are now ready to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Testing formally (1.1) against v , we get

$$(4.1) \quad \frac{1}{2} \frac{d}{dt} (\|\nabla v\|^2 + \alpha^2 \|\partial_1 \nabla v\|^2) + \nu (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) \leq |(g, v)|.$$

Since $|(g, v)| \leq \lambda_1^{-1} \|g\| \cdot \|\Delta v\| \leq \frac{1}{2\nu\lambda_1^2} \|g\|^2 + \frac{\nu}{2} \|\Delta v\|^2$, we deduce

$$\frac{d}{dt} (\|\nabla v\|^2 + \alpha^2 \|\partial_1 \nabla v\|^2) + \nu (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) \leq \frac{1}{\nu\lambda_1^2} \|g\|^2,$$

which implies

$$\begin{aligned} \|\nabla v(t)\|^2 + \alpha^2 \|\partial_1 \nabla v(t)\|^2 + \nu \int_0^t (\|\Delta v(s)\|^2 + \alpha^2 \|\partial_1 \Delta v(s)\|^2) ds \\ \leq \frac{1}{\nu\lambda_1^2} \|g\|^2 t + \|\nabla v(0)\|^2 + \alpha^2 \|\partial_1 \nabla v(0)\|^2, \end{aligned}$$

so that $v \in L_{\text{loc}}^\infty(0, \infty; H_0^{2,h}) \cap L_{\text{loc}}^2(0, \infty; H_0^{2,h} \cap H^{3,h})$.

Multiplying (1.1) against v_t and integrating over Ω , we get

$$(4.2) \quad \begin{aligned} \frac{\nu}{2} \frac{d}{dt} (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) + \|\nabla v_t\|^2 + \alpha^2 \|\partial_1 \nabla v_t\|^2 \\ \leq |(g, v_t)| + |(B(v, v), v_t)|. \end{aligned}$$

We have $|(g, v_t)| \leq \lambda_1^{-1/2} \|g\| \cdot \|\nabla v_t\| \leq \frac{\|g\|^2}{\lambda_1} + \frac{\|\nabla v_t\|^2}{4}$ and, thanks to the Hölder, the Gagliardo–Nirenberg and the Young inequalities, we have also

$$\begin{aligned} |(B(v, v), v_t)| &\leq \|\partial_1 v\|_{L^\infty} \|\Delta v\| \|\partial_2 v_t\| + \|\partial_2 v\|_{L^4} \|\Delta v\| \|\partial_1 v_t\|_{L^4} \\ &\leq \|\partial_1 \Delta v\| \|\Delta v\| \|\nabla v_t\| + C \|\Delta v\|^2 \|\partial_1 \nabla v_t\| \\ &\leq \frac{\|\nabla v_t\|^2}{4} + \frac{\alpha^2}{2} \|\partial_1 \nabla v_t\|^2 + C\nu (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) \|\Delta v\|^2, \end{aligned}$$

for a suitable constant $C = C(\lambda_1, \alpha, \nu) > 0$. Plugging this estimate in (4.2), we obtain

$$\begin{aligned} \nu \frac{d}{dt} (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) + \|\nabla v_t\|^2 + \alpha^2 \|\partial_1 \nabla v_t\|^2 \\ \leq C \|g\|^2 + C\nu (\|\Delta v\|^2 + \alpha^2 \|\partial_1 \Delta v\|^2) \|\Delta v\|^2. \end{aligned}$$

Since we have already proved that $v \in L_{\text{loc}}^2(H^2)$, an application of the Grönwall lemma gives the claimed regularity of v (here we use the full regularity of v_0), and consequently by the previous inequality, the regularity of v_t .

Lastly, notice that the proof of the uniqueness of weak solutions is quite standard and very similar to the proof of uniqueness for the case of a bounded domain (mainly because of the validity of the Poincaré inequality) and this last part of the proof is left to the reader. \square

5. WEAK SOLUTIONS IN ANISOTROPIC WEIGHTED SOBOLEV SPACES

In what follows we prove the main result of the paper, i.e. Theorem 3.1. Let us consider the weight function $\phi = (1 + |x_1|^3 + |x_2|^2)^\gamma$, $\gamma > 0$, introduced in Subsection 2.1, and the approximating function $\varphi = \varphi(x, \epsilon, \rho, \gamma)$ with $\psi = \varphi^{\frac{1}{2}}$.

We state the following technical lemma.

Lemma 5.1. *Under the assumption $\gamma \leq 2/3$, setting $\psi = \varphi^{1/2}$, then it holds true that*

$$(5.1) \quad |\partial^\beta \psi^2| \leq C \epsilon^{|\beta|} \psi, \quad \beta = (\beta_1, \beta_2), \quad 0 < |\beta| \leq 3, \quad \beta_2 \leq 2.$$

The precise construction of the weight function φ and the proof of this lemma are postponed to Appendix A. For the remainder of the paper we always assume that $\gamma \leq 2/3$.

In the proof of existence in weighted spaces, we will use the following result (see also [20, Theorem 2.2] and [2]) to overcome the difficulties arising because of the unboundedness of the strip-like region Ω .

Lemma 5.2 (Corollary 2.34, [23]). *Let Θ be a bounded set of \mathbb{R}^d , $X \subset E$ Banach spaces with compact injection. Consider $1 \leq p < q \leq +\infty$. Suppose that $\mathcal{F} \subset L^p(\Theta; E)$ satisfies*

- (i) $\forall \mathcal{W} \subset \subset \Theta, \limsup_{k \rightarrow 0} \sup_{f \in \mathcal{F}} \|\tau_k f - f\|_{L^p(\mathcal{W}; E)} = 0$ (where $\tau_k f$ is the translation given by $\tau_k f(x) = f(x + k)$),
- (ii) \mathcal{F} is bounded in $L^q(\Theta; E) \cap L^1(\Theta, X)$.

Then, \mathcal{F} is precompact in $L^p(\Theta; E)$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Since $H_\gamma^{3,h} \cap H_0^{2,h}$ is separable and the set $\mathcal{V} = \{v \in C_0^\infty(\Omega) : \nabla \cdot v = 0\}$ is dense in $H_\gamma^{3,h} \cap H_0^{2,h}$, there exists a sequence of linearly independent elements $\{w_1, w_2, \dots\} \subset \mathcal{V}$ which is complete in $H_\gamma^{3,h} \cap H_0^{2,h}$. Denote $H_m := \text{span}\{w_j\}_{j=1, \dots, m}$ and consider the projector $P_m(v) = \sum_{j=1}^m (v, w_j) w_j$. A function

$$v^m = \sum_{j=1}^m a_j^m(t) w_j(x)$$

is an m -approximate solution of Equation (1.1) if

$$\begin{aligned} & (\nabla v_t^m, \nabla w_j) + \alpha^2 (\partial_1 \nabla v_t^m, \partial_1 \nabla w_j) + \nu (\Delta v^m, \Delta w_j) + \nu \alpha^2 (\partial_1 \Delta v^m, \partial_1 \Delta w_j) \\ & = (B(v^m, v^m), w_j) - (g, w_j) \end{aligned}$$

for every $j = 1, \dots, m$. The existence of solutions is guaranteed by the Peano theorem.

We split the proof in a number of steps.

- (1) We establish a priori estimates for $\{v^m\}$ in the space $L_{\text{loc}}^\infty(0, \infty; H_\gamma^{2,h}) \cap L_{\text{loc}}^2(0, \infty; H_\gamma^{3,h})$.
- (2) We show that $\{v^m\}$ satisfies condition (ii) of Lemma 5.2: $\mathcal{F} := \{v^m|_{\mathcal{O}}\}$ is bounded in $L^\infty(\tau, T; H_\gamma^{2,h}(\mathcal{O})) \cap L^1(\tau, T; H_\gamma^{3,h}(\mathcal{O}))$, where \mathcal{O} is any open subset in Ω and $\Theta := (\tau, T)$.
- (3) We show that $\{v^m\}$ satisfies condition (i) of Lemma 5.2:

$$\lim_{k \rightarrow 0} \sup_{m \in N} \|\tau_k v^m - v^m\|_{L^2(\tau, T-k; H_\gamma^{2,h}(\mathcal{O}))} = 0.$$
- (4) We apply Lemma 5.2 and extract a subsequence still denoted by $v^m|_{\mathcal{O}}$ converging to some v in $L^2(\tau, T; H_\gamma^{2,h}(\mathcal{O}))$. In particular, we observe that the relations in the previous two points are uniform with respect to \mathcal{O} .
- (5) The limiting function v is a weak solution.
- (6) By interpolation, we obtain the time continuity of v with values in $H_\gamma^{1,h}(\Omega)$.

STEP 1: *Establishing a priori estimates in $L_{\text{loc}}^\infty(0, \infty; H_\gamma^{2,h}) \cap L_{\text{loc}}^2(0, \infty; H_\gamma^{3,h})$.*

Here, we proceed again formally by dealing with v and the equation satisfied by it. However, the a priori estimates that we are about to derive can be rigorously justified. Indeed, a rigorous proof uses v^m instead of v and w_j as test functions, as it will be made in the second and third steps below.

We multiply equation (1.1) by $v\psi^2$ in $L^2(\Omega)$ and use integration by parts to get

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left(\|\psi \nabla v\|^2 + \alpha^2 \|\psi \partial_1 \nabla v\|^2 \right) + \nu \|\psi \Delta v\|^2 + \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 \\
 &= (B(v, v) - g, v\psi^2) - \int \nabla v_t v \nabla \psi^2 + \alpha^2 \int \partial_1^2 \nabla v_t v \nabla \psi^2 \\
 & \quad - \alpha^2 \int \partial_1 \nabla v_t \nabla v \partial_1 \psi^2 - \nu \int \Delta v v \Delta \psi^2 - 2\nu \int \Delta v \nabla v \nabla \psi^2 \\
 (5.2) \quad & \quad - 2\nu \alpha^2 \int \partial_1 \Delta v \partial_1 \nabla v \nabla \psi^2 - \nu \alpha^2 \int \partial_1 \Delta v \partial_1 v \Delta \psi^2 - \nu \alpha^2 \int \partial_1 \Delta v \Delta v \partial_1 \psi^2 \\
 & \quad - 2\nu \alpha^2 \int \partial_1 \Delta v \nabla v \partial_1 \nabla \psi^2 - \nu \alpha^2 \int \partial_1 \Delta v v \partial_1 \Delta \psi^2 \\
 &= (B(v, v) - g, v\psi^2) + \sum_{j=1}^{10} L_j.
 \end{aligned}$$

In particular, we have used that

$$\begin{aligned}
 & - \int \partial_1^2 \Delta^2 v v \psi^2 = \int \partial_1 \Delta^2 v \partial_1 v \psi^2 + \int \partial_1 \Delta^2 v v \partial_1 \psi^2 \\
 &= - \int \partial_1 \nabla \Delta v \partial_1 \nabla v \psi^2 - \int \partial_1 \nabla \Delta v \partial_1 v \nabla \psi^2 \\
 & \quad - \int \partial_1 \nabla \Delta v \nabla v \partial_1 \psi^2 - \int \partial_1 \nabla \Delta v v \partial_1 \nabla \psi^2 \\
 &= \int \partial_1 \Delta v \partial_1 \Delta v \psi^2 + 2 \int \partial_1 \Delta v \partial_1 \nabla v \nabla \psi^2 + \int \partial_1 \Delta v \partial_1 v \Delta \psi^2 \\
 & \quad + \int \partial_1 \Delta v \Delta v \partial_1 \psi^2 + 2 \int \partial_1 \Delta v \nabla v \partial_1 \nabla \psi^2 + \int \partial_1 \Delta v v \partial_1 \Delta \psi^2
 \end{aligned}$$

and noticed that all boundary terms are zero.

Proceeding as in [14], we immediately have the existence of a constant $C = C(\lambda_1, \nu, \alpha) > 0$ such that

$$\begin{aligned}
 (B(v, v), v\psi^2) &= \int [\partial_1 v \Delta v \partial_2 (v\psi^2) - \partial_2 v \Delta v \partial_1 (v\psi^2)] \\
 &= \int [\partial_1 v \Delta v v \partial_2 \psi^2 - \partial_2 v \Delta v v \partial_1 \psi^2] \\
 &\leq 2\|v\|_{L^\infty} \|\psi \Delta v\| \|\psi \nabla v\| \\
 &\leq \epsilon \nu \|\psi \Delta v\|^2 + C\epsilon \|\psi \nabla v\|^2
 \end{aligned}
 \tag{5.3}$$

and

$$|(g, v\psi^2)| \leq C\|\psi g\|^2 + \epsilon \nu \|\psi \Delta v\|^2,$$

where to control v in L^∞ -norm we use Agmon's inequality and the regularity provided by Theorem 1.1.

Then, we estimate the terms L_i , $i = 1, \dots, 10$. Let us start with L_1 , to get

$$\begin{aligned}
 |L_1| &\leq \left| \int \nabla v_t v \nabla \psi^2 \right| \\
 &\leq \left| \int v_t \nabla v \nabla \psi^2 \right| + \left| \int v_t v \Delta \psi^2 \right| \\
 &\leq C\epsilon \int |\psi \nabla v| |v_t| + C\epsilon^2 \int |\psi v| |v_t| \\
 &\leq C\epsilon (\|\psi \nabla v\|^2 + \|v_t\|^2)
 \end{aligned}
 \tag{5.4}$$

where we used the relation (5.1). Similarly, we also have that

$$\begin{aligned}
 |L_2| &\leq \alpha^2 \left| \int \partial_1 \nabla v_t \partial_1 v \nabla \psi^2 \right| + \alpha^2 \left| \int \partial_1 \nabla v_t v \partial_1 \nabla \psi^2 \right| \\
 &\leq \alpha^2 \left| \int \partial_1 v_t \partial_1 \nabla v \nabla \psi^2 \right| + \alpha^2 \left| \int \partial_1 v_t \partial_1 v \Delta \psi^2 \right| \\
 &\quad + \alpha^2 \left| \int \partial_1 v_t \nabla v \partial_1 \nabla \psi^2 \right| + \alpha^2 \left| \int \partial_1 v_t v \partial_1 \Delta \psi^2 \right| \\
 &\leq \epsilon \alpha^2 C \|\partial_1 v_t\|^2 + \epsilon C \alpha^2 (\|\psi \partial_1 \nabla v\|^2 + \|\psi \nabla v\|^2).
 \end{aligned}$$

Also in this case we conclude by using (5.1). For the term L_3 we have

$$\begin{aligned}
 |L_3| &\leq \alpha^2 \left| \int \partial_1 \nabla v_t \nabla v \partial_1 \psi^2 \right| \\
 &\leq \alpha^2 \left| \int \partial_1 v_t \Delta v \partial_1 \psi^2 \right| + \left| \int \partial_1 v_t \nabla v \partial_1 \nabla \psi^2 \right| \\
 &\leq \epsilon \alpha^2 C \|\partial_1 v_t\|^2 + \epsilon \alpha^2 C (\nu \|\psi \Delta v\|^2 + \|\psi \nabla v\|^2).
 \end{aligned}$$

Next, for the terms L_4 and L_5 , using the same inequalities we get

$$\begin{aligned}
 |L_4| &\leq 2\nu \int |\Delta v| |v| |\Delta \psi^2| \leq 2\nu \epsilon^2 \int |\psi \Delta v| |\psi v| \\
 &\leq \epsilon^2 \nu \|\psi \Delta v\|^2 + C\epsilon^2 \|\psi v\|^2,
 \end{aligned}$$

and

$$\begin{aligned} |L_5| &\leq 2\nu \left| \int \nabla v \Delta v \nabla \psi^2 \right| \\ &\leq \epsilon \nu \|\psi \Delta v\|^2 + \epsilon \nu C \|\psi \nabla v\|^2. \end{aligned}$$

Again, for the terms L_6 , L_7 and L_8 we have

$$\begin{aligned} |L_6| &\leq \nu \alpha^2 \left| \int \partial_1 \nabla v \partial_1 \Delta v \nabla \psi^2 \right| + \nu \alpha^2 \left| \int \partial_1 \nabla v \partial_1 \nabla v \Delta \psi^2 \right| \\ &\leq \epsilon \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 + C \epsilon \|\psi \partial_1 \nabla v\|^2, \end{aligned}$$

$$\begin{aligned} |L_7| &\leq \nu \alpha^2 \left| \int \partial_1 \Delta v \partial_1 v \Delta \psi^2 \right| \leq \nu \alpha^2 \epsilon^2 \int |\psi \partial_1 \Delta v| |\psi \partial_1 v| \\ &\leq \epsilon^2 \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 + C \epsilon^2 \alpha^2 \nu \|\psi \nabla v\|^2 \end{aligned}$$

and

$$\begin{aligned} |L_8| &\leq \nu \alpha^2 \left| \int \partial_1 \Delta v \Delta v \partial_1 \psi^2 \right| \leq \nu \alpha^2 \epsilon \int |\psi \partial_1 \Delta v| |\psi \Delta v| \\ &\leq \frac{\nu \alpha^2 \epsilon}{2} \|\psi \partial_1 \Delta v\|^2 + \frac{\nu \alpha^2 \epsilon}{2} \|\psi \Delta v\|^2. \end{aligned}$$

Finally, for the last two terms, exploiting similar estimates, we get

$$\begin{aligned} |L_9| &\leq \nu \alpha^2 \epsilon^2 \int |\psi \partial_1 \Delta v| |\psi \nabla v| \\ &\leq \epsilon \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 + C \epsilon^2 \psi \|\nabla v\|^2 \end{aligned}$$

and

$$\begin{aligned} (5.5) \quad |L_{10}| &\leq \nu \alpha^2 \int |\partial_1 \Delta v| |v| |\partial_1 \Delta \psi^2| \leq \nu \alpha^3 \epsilon^2 \int |\psi \partial_1 \nabla \Delta v| |\psi v| \\ &\leq \epsilon^3 \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 + C \epsilon^3 \|\psi v\|^2. \end{aligned}$$

Using (5.2) along with the estimates (5.3)–(5.5) we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\psi \nabla v\|^2 + \alpha^2 \|\psi \partial_1 \nabla v\|^2 \right) + \nu \|\psi \Delta v\|^2 + \nu \alpha^2 \|\psi \partial_1 \Delta v\|^2 \\ &\leq \epsilon C \|v_t\|^2 + \epsilon \alpha^2 C \|\partial_1 v_t\|^2 + \epsilon \nu C \|\psi \Delta v\|^2 \\ &\quad + \epsilon \nu \alpha^2 C \|\psi \partial_1 \Delta v\|^2 + \epsilon C \|\psi \nabla v\|^2 + C \|\psi g\|^2. \end{aligned}$$

Using the control on $\|v_t\|$ and $\|\partial_1 v_t\|$ provided by Theorem 1.1 together with the Grönwall inequality, we get the claimed regularity on v , i.e. $v \in L_{\text{loc}}^\infty(0, \infty; H_\gamma^{2,h}) \cap L_{\text{loc}}^2(0, \infty; H_\gamma^{3,h})$. This concludes STEP 1.

Before proceeding with the next steps, we open a parenthesis to outline the scheme behind the remaining part of the proof. Until now, we have used v in place of v^m for a matter of convenience; however, in view of extracting a convergent subsequence of $\{v^m\}$, here below we will employ this latter notation. From the above estimates, we can extract a subsequence of $\{v^m\}$, still denoted by $\{v^m\}$, such that

$$\begin{aligned} v^m &\rightharpoonup \tilde{v} \text{ weak-star in } L^\infty(\tau, T; H_\gamma^{2,h}(\Omega)), \\ v^m &\rightharpoonup \tilde{v} \text{ weak in } L^2(\tau, T; H_\gamma^{3,h}(\Omega)). \end{aligned}$$

Moreover, as a consequence of the estimates in the proof of Theorem 1.1 we also have that

$$(5.6) \quad v^m \rightarrow \tilde{v} \text{ strong in } L^2(\tau, T; H^{2,h}_\gamma(\Omega)).$$

To conclude our argument, obtaining that $\{v^m\}$ is relatively compact in $L^2(\tau, T; H^{2,h}_\gamma(\Omega))$, we would need some control on dv^m/dt .

When it is possible to choose a special basis $w_j \in C_0^\infty(\Omega)$ to generate the Galerkin elements $v^m(x, t) = \sum_{j=1}^m a_j^m(t)w_j(x)$, $m \in \mathbb{N}$, such that a uniform control on $\|dv^m/dt\|_{L^2(\tau, T; H^{1,h}_\gamma(\Omega))}$ holds true, this is enough to use a compactness result à la Aubin–Lions to get the existence of a subsequence such that $v^m \rightarrow \tilde{v}$ in $L^2(\tau, T; H^{2,h}_\gamma(\Omega))$, and even more.

Here, using Lemma 5.2, we obtain a similar result but not on the whole domain Ω . Actually, what we are going to prove is the following: for any bounded open set $\mathcal{O} \subset \Omega$, there exists a subsequence of $\{v^m\}$ (depending on \mathcal{O} and relabeled $\{v^m|_{\mathcal{O}}\}$) satisfying

$$(5.7) \quad v^m|_{\mathcal{O}} \rightarrow v|_{\mathcal{O}} \text{ in } L^2(\tau, T; H^{1,h}_\gamma(\mathcal{O})).$$

Since we also have that $\{v^m\}$ is weakly convergent to \tilde{v} in $L^2(\tau, T; H^{3,h}_\gamma)$, due to the uniqueness of the limit it follows that $(\tilde{v})|_{\mathcal{O}} = v|_{\mathcal{O}}$ for every bounded subset $\mathcal{O} \subset \Omega$. This fact along with (5.7) will be enough to prove that \tilde{v} is a weak solution to (1.1) defined in $L^\infty_{\text{loc}}(0, \infty; H^{2,h}_\gamma) \cap L^2_{\text{loc}}(0, \infty; H^{3,h}_\gamma)$. Indeed, to conclude our analysis on $\Omega \times (\tau, T)$, and to prove that the weak formulation for v^m is stable when $m \rightarrow +\infty$, we consider a proper family of test functions with separate variables and bounded supports (see, e.g., [2]). Let $\{w_j\}_{j=1,\dots,m}$ be the basis of the space H_m approximating $H^{3,h}_\gamma \cap H^2_0$, for $m \in \mathbb{N}$. Let $\sigma = \sigma(t)$ be a continuously differentiable function on $[\tau, T]$ with $\sigma(T) = 0$. Then, we set the following weak formulation (where $w_j(x)\sigma(t)$ are the tests) on $\Omega \times (\tau, T)$:

$$\begin{aligned} & \int_\tau^T (\psi \nabla v_t^m, \psi \nabla w_j) \sigma dt + \alpha^2 \int_\tau^T (\psi \partial_1 \nabla v_t^m, \psi \partial_1 \nabla w_j) \sigma dt + \nu \int_\tau^T (\psi \Delta v^m, \psi \Delta w_j) \sigma dt \\ & + \nu \alpha^2 \int_\tau^T (\psi \partial_1 \Delta v^m, \psi \partial_1 \Delta w_j) \sigma dt + \int_\tau^T (\psi B(v^m, v^m), \psi w_j) \sigma dt \\ & = \int_\tau^T (\psi g, \psi w_j) \sigma dt \end{aligned}$$

for all $j = 1, \dots, m$. Using Lemma 5.2 (the intersection $\text{supp } w_j \cap \Omega$ is bounded) we will prove that the above relation passes to the limit as $m \rightarrow +\infty$.

To proceed to the next steps, and prove that Lemma 5.2 applies to our case, we set $X = H^{3,h}_\gamma(\mathcal{O})$, $E = H^{2,h}_\gamma(\mathcal{O})$, where \mathcal{O} is any open set included in Ω . Also, we choose $p = 2$, $q = +\infty$ and, as already mentioned, we denote by $\Theta = (\tau, T) \subseteq \mathbb{R}$ the time interval, and by $\mathcal{F} = \{v^m|_{\mathcal{O}}\}$ the approximating sequence.

STEP 2: *The approximating sequence $\{v^m\}$ satisfies condition (ii) in Lemma 5.2.* The boundedness of v^m in $L^\infty(\tau, T; H^{2,h}_\gamma(\mathcal{O})) \cap L^1(\tau, T; H^{3,h}_\gamma(\mathcal{O}))$, $\mathcal{O} \subset \Omega$ open and bounded, follows directly from the boundedness of v^m in $L^\infty_{\text{loc}}(0, \infty; H^{2,h}_\gamma) \cap L^2_{\text{loc}}(0, \infty; H^{3,h}_\gamma)$ proved in STEP 1. This concludes STEP 2.

STEP 3: *The approximating sequence $\{v^m\}$ satisfies condition (i) in Lemma 5.2.*

First, we will prove that $\{v^m|_{\mathcal{O}}\}$ is relatively compact in $L^2(\tau, T; H^{2,h}_\gamma(\mathcal{O}))$ for all bounded subsets $\mathcal{O} \subset \Omega$ by using Lemma 5.2. We only have to check that $\{v^m\}$

satisfies condition (i) in Lemma 5.2, i.e.,

$$\lim_{k \rightarrow 0} \sup_{m \in \mathbb{N}} \|\tau_k v^m - v^m\|_{L^2(\tau, T-k; H_\gamma^{2,h}(\mathcal{O}))} = 0.$$

Also here, to keep the notation as compact as possible, we write v in place of v^m . Consider $k > 0$ arbitrarily small and set $V_k(t) := (v(t+k) - v(t))$. We take the product of (1.1) against $-\psi^2 w_j$, integrate in time over $(t, t+k) \subset (\tau, T)$; subsequently, we multiply it by

$$a_j^m(t+k) - a_j^m(t),$$

and by summing over j , we reach (here below, we reintroduce the dx in the space-depending integrals)

$$\begin{aligned} & \|\psi \nabla V_k(t)\|^2 + \alpha^2 \|\psi \partial_1 \nabla V_k(t)\|^2 \\ &= -\nu \int_t^{t+k} \int \psi \Delta v \psi \Delta V_k(t) dx ds - \nu \alpha^2 \int_t^{t+k} \int \psi \partial_1 \Delta v \partial_1 \Delta V_k(t) dx ds \\ & \quad + \int_t^{t+k} (B(v, v) - g, V_k(t) \psi^2) ds - \int \nabla V_k(t) V_k(t) \nabla \psi^2 dx \\ & \quad + \alpha^2 \int \partial_1^2 \nabla V_k(t) V_k(t) \nabla \psi^2 dx - \alpha^2 \int \partial_1 \nabla V_k(t) \nabla V_k(t) \partial_1 \psi^2 dx \\ & \quad - \nu \int_t^{t+k} \int \Delta v V_k(t) \Delta \psi^2 dx ds - 2\nu \int_t^{t+k} \int \Delta v \nabla V_k(t) \nabla \psi^2 dx ds \\ (5.8) \quad & - 2\nu \alpha^2 \int_t^{t+k} \int \partial_1 \Delta v \partial_1 \nabla V_k(t) \nabla \psi^2 \\ & \quad - \nu \alpha^2 \int_t^{t+k} \int \partial_1 \Delta v \partial_1 V_k(t) \Delta \psi^2 dx ds \\ & \quad - \nu \alpha^2 \int_t^{t+k} \int \partial_1 \Delta v \Delta V_k(t) \partial_1 \psi^2 dx ds \\ & \quad - 2\nu \alpha^2 \int_t^{t+k} \int \partial_1 \Delta v \nabla V_k(t) \partial_1 \nabla \psi^2 dx ds \\ & \quad - \nu \alpha^2 \int_t^{t+k} \int \partial_1 \Delta v V_k(t) \partial_1 \Delta \psi^2 dx ds \end{aligned}$$

from which, integrating on $(\tau, T-k)$ in dt , we get

$$\begin{aligned} & \int_\tau^{T-k} \|\psi \nabla V_k(t)\|^2 dt + \alpha^2 \int_\tau^{T-k} \|\psi \partial_1 \nabla V_k(t)\|^2 dt \\ &= -\nu \int_\tau^{T-k} \int_t^{t+k} \int \psi \Delta v \psi \Delta V_k(t) dx ds dt \\ (5.9) \quad & - \nu \alpha^2 \int_\tau^{T-k} \int_t^{t+k} \int \psi \partial_1 \Delta v \partial_1 \Delta V_k(t) dx ds dt \\ & \quad + \int_\tau^{T-k} \int_t^{t+k} (B(v, v) - g, V_k(t) \psi^2) ds dt + \int_\tau^{T-k} \sum_{j=1}^{10} J_j. \end{aligned}$$

For the terms in the right-hand side of the above equality, exploiting the Fubini's theorem along with the properties of the following functions

$$\bar{s} = \begin{cases} \tau & \text{if } s \leq \tau \\ s & \text{if } \tau < s \leq T-k \\ T-k & \text{if } s > T-k \end{cases} \quad \text{and} \quad \overline{s-k} = \begin{cases} \tau & \text{if } s-k \leq \tau \\ s-k & \text{if } \tau < s-k \leq T-k \\ T-k & \text{if } s-k > T-k \end{cases}$$

which are used to change the order of integration, we get

$$\begin{aligned} & \nu \left| \int_{\tau}^{T-k} \int_t^{t+k} \int \psi \Delta v(s) \psi \Delta V_k(t) dx ds dt \right| \\ & \leq \nu \int_{\tau}^{T-k} \int_t^{t+k} \|\psi \Delta v(s)\| \|\psi \Delta V_k(t)\| ds dt \\ & \leq \nu \int_{\tau}^{T-k} \|\psi \Delta V_k(t)\| \int_t^{t+k} \|\psi \Delta v(s)\| ds dt \\ & \leq \nu \int_{\tau}^T \|\psi \Delta v(s)\| \int_{s-k}^{\bar{s}} \|\psi \Delta V_k(t)\| dt ds \\ & \leq \nu \int_{\tau}^T \|\psi \Delta v(s)\| \left(\int_{s-k}^{\bar{s}} 1 dt \right)^{1/2} \left(\int_{s-k}^{\bar{s}} \|\psi \Delta V_k(t)\|^2 dt \right)^{1/2} ds \\ & \leq 2\nu k^{1/2} (T-\tau)^{1/2} \|\psi \Delta v\|_{L^2(\tau, T; L^2)} \left(\int_{\tau}^T \|\psi \Delta v(s)\|^2 ds \right)^{1/2} \\ & \leq 2\nu k^{1/2} (T-\tau)^{1/2} \|\psi \Delta v\|_{L^2(\tau, T; L^2)}^2 \\ & \leq C\nu k^{1/2} (T-\tau)^{1/2}. \end{aligned}$$

With similar computations, we also obtain that

$$\begin{aligned} & \nu \alpha^2 \left| \int_{\tau}^{T-k} \int_t^{t+k} \int \psi \partial_1 \Delta v \partial_1 \Delta V_k(t) dx ds dt \right| \\ & \leq 2\nu \alpha^2 k^{1/2} \|\psi \partial_1 \Delta v\|_{L^2(\tau, T; L^2)}^2 (T-\tau)^{1/2} \\ & \leq C\nu \alpha^2 k^{1/2} (T-\tau)^{1/2}. \end{aligned}$$

Now, by exploiting (2.9), we have that

$$\begin{aligned} & \left| \int_{\tau}^{T-k} \int_t^{t+k} (B(v, v), \psi^2 V_k(t)) ds dt \right| \\ & \leq \left| \int_{\tau}^{T-k} \int_t^{t+k} \int [\partial_1 v \Delta V_k(t) \partial_2 \psi^2 - \partial_2 v \Delta V_k(t) \partial_1 \psi^2] dx ds dt \right| \\ & \quad + \left| \int_{\tau}^{T-k} \int_t^{t+k} \int [\partial_1 v \Delta v \psi^2 \partial_2 V_k(t) - \partial_2 v \Delta v \psi^2 \partial_1 V_k(t)] dx ds dt \right| \\ & \leq C\epsilon \int_{\tau}^{T-k} \int_t^{t+k} \|\Delta v\| (\|\nabla \partial_1 v\| + \|\nabla \partial_2 v\|) (\|\psi \nabla V_k(t)\| + \epsilon \|\psi V_k(t)\|) ds dt \\ & \quad + C \int_{\tau}^{T-k} \int_t^{t+k} \|\psi \Delta v\| (\|\nabla \partial_1 v\| + \|\nabla \partial_2 v\|) (\|\psi \Delta V_k(t)\| + \epsilon \|\psi \nabla V_k(t)\|) ds dt \\ & \leq Ck^{1/2} (T-\tau)^{1/2} (\|v\|_{L^2(\tau, T; H_1^1)}^2 + \|v\|_{L^2(\tau, T; H_2^1)}^2) \\ & \leq Ck^{1/2} (T-\tau)^{1/2}. \end{aligned}$$

Now, we estimate the terms J_i , $i = 1, \dots, 10$. Let us start with J_1 to get

$$(5.10) \quad \begin{aligned} \int_{\tau}^{T-k} |J_1| dt &\leq \epsilon \int_{\tau}^{T-k} \int |\psi \nabla V_k(t)| |\psi V_k(t)| dx dt \\ &\leq \epsilon \sqrt{\frac{2}{\lambda_1}} \int_{\tau}^{T-k} \|\psi \nabla V_k(t)\|^2 dt. \end{aligned}$$

For the terms J_2 and J_3 we have that

$$(5.11) \quad \begin{aligned} \int_{\tau}^{T-k} |J_2| dt &\leq \alpha^2 \epsilon \int_{\tau}^{T-k} \int |\psi \partial_1 \nabla V_k(t)| |\psi \partial_1 V_k(t)| dt \\ &\quad + \alpha^2 \epsilon \int_{\tau}^{T-k} \int |\psi \partial_1 \nabla V_k(t)| |\psi V_k(t)| dt \\ &\leq C \alpha^2 \epsilon \int_{\tau}^{T-k} \|\psi \partial_1 \nabla V_k(t)\|^2 dt \\ &\quad + C \alpha^2 \epsilon \int_{\tau}^{T-k} \|\psi \partial_1 \nabla V_k(t)\| \|\psi \nabla V_k(t)\| dt \\ &\leq C \alpha^2 \epsilon \int_{\tau}^{T-k} \|\psi \partial_1 \nabla V_k(t)\|^2 dt \\ &\quad + C \alpha^2 \epsilon (T - \tau)^{1/2} \|\psi \nabla v\|_{L^\infty(\tau, T; L^2)} \|\psi \nabla v\|_{L^2(\tau, T; L^2)} \end{aligned}$$

and that

$$(5.12) \quad \begin{aligned} \int_{\tau}^{T-k} |J_3| dt &\leq \alpha^2 \epsilon \int_{\tau}^{T-k} \int |\psi \partial_1 \nabla V_k(t)| |\psi \nabla V_k(t)| dx dt \\ &\leq C \alpha^2 \epsilon \int_{\tau}^{T-k} \|\psi \partial_1 \nabla V_k(t)\| \|\psi \nabla V_k(t)\| dt \\ &\leq C \alpha^2 \epsilon \int_{\tau}^{T-k} \|\psi \partial_1 \nabla V_k(t)\|^2 dt \\ &\quad + C \alpha^2 \epsilon (T - \tau)^{1/2} \|\psi \nabla v\|_{L^\infty(\tau, T; L^2)} \|\psi \nabla v\|_{L^2(\tau, T; L^2)}. \end{aligned}$$

Next, for the terms J_4 and J_5 we have

$$\begin{aligned} \int_{\tau}^{T-k} |J_4| dt &\leq \nu \int_{\tau}^{T-k} \int_t^{t+k} \int \Delta v V_k(t) \Delta \psi^2 dx ds dt \\ &\leq \epsilon \nu \int_{\tau}^{T-k} \int_t^{t+k} \|\Delta v(s)\| \|\psi V_k(t)\| ds dt \\ &\leq C \epsilon \nu k^{1/2} (T - \tau)^{1/2} \|\psi v\|_{L^2(\tau, T; L^2)}, \end{aligned}$$

where we used again (5.1), and

$$\begin{aligned} \int_{\tau}^{T-k} |J_5| dt &\leq 2\epsilon \nu \int_{\tau}^{T-k} \int_t^{t+k} \|\psi \Delta v(s)\| \|\psi \nabla V_k(t)\| ds dt \\ &\leq 4\nu \epsilon k^{1/2} \|v\|_{L^2(\tau, T; H_\gamma^1)} \int_{\tau}^T \|\psi \Delta v(s)\| ds \\ &\leq C \nu \epsilon k^{1/2} (T - \tau)^{1/2} \|v\|_{L^2(\tau, T; H_\gamma^2)}^2. \end{aligned}$$

J_6 is estimated as follows:

$$\begin{aligned} \int_{\tau}^{T-k} |J_6| dt &\leq 2\nu\alpha^2\epsilon \int_{\tau}^{T-k} \int_t^{t+k} \|\psi\partial_1\Delta v\| \|\psi\partial_1\nabla V_k(t)\| ds dt \\ &\leq 4\nu\epsilon k^{1/2}\alpha^2\|v\|_{L^2(\tau,T;H_{\gamma}^{2,h})} \int_{\tau}^T \|\psi\partial_1\Delta v(s)\| ds \\ &\leq C\nu\epsilon k^{1/2}(T-\tau)^{1/2}\alpha^2\|v\|_{L^2(\tau,T;H_{\gamma}^{3,h})}^2. \end{aligned}$$

In a very similar way we also get

$$\begin{aligned} \int_{\tau}^{T-k} |J_7| dt &\leq \nu\epsilon\alpha^2 C k^{1/2} \|v\|_{L^2(\tau,T;H_{\gamma}^{1,h})} \int_{\tau}^T \|\psi\partial_1\Delta v(s)\| ds, \\ \int_{\tau}^{T-k} |J_8| dt &\leq \nu\alpha^2\epsilon C k^{1/2} \|v\|_{L^2(\tau,T;H_{\gamma}^2)} \int_{\tau}^T \|\psi\partial_1\Delta v(s)\| ds, \\ \int_{\tau}^{T-k} |J_9| dt &\leq \nu\alpha^2\epsilon C k^{1/2} \|v\|_{L^2(\tau,T;H_{\gamma}^1)} \int_{\tau}^T \|\psi\partial_1\Delta v(s)\| ds, \end{aligned}$$

and

$$\int_{\tau}^{T-k} |J_{10}| dt \leq \nu\alpha^2\epsilon C k^{1/2} \|v\|_{L^2(\tau,T;L_{\gamma}^2)} \int_{\tau}^T \|\psi\partial_1\Delta v(s)\| ds.$$

Whence, for $i = 7, 8, 9, 10$, we have that

$$(5.13) \quad \int_{\tau}^{T-k} |J_i| dt \leq C k^{1/2} (T-\tau)^{1/2} \|v\|_{L^2(\tau,T;H_{\gamma}^{3,h})}^2.$$

To conclude we reabsorb the terms (5.10), (5.11) and (5.12) in the left-hand side of (5.8). Then, using standard manipulations along with the above estimates we get

$$\int_{\tau}^{T-k} \|\psi\nabla V_k(t)\|^2 dt + \alpha^2 \int_{\tau}^{T-k} \|\psi\partial_1\nabla V_k(t)\|^2 dt \leq C(T-\tau)^{1/2} k^{1/2} \rightarrow 0 \text{ as } k \rightarrow 0;$$

this concludes STEP 3.

STEP 4: *Application of Lemma 5.2 to $\{v^m|_{\mathcal{O}}\}$.*

By Lemma 5.2, we deduce that $\{v^m|_{\mathcal{O}}\}$ is relatively compact in the space $L^2(\tau, T; H_{\gamma}^{2,h}(\mathcal{O}))$, uniformly with respect to \mathcal{O} , and we can extract a subsequence, still denoted by $\{v^m|_{\mathcal{O}}\}$, that strongly converges in $L^2(\tau, T; H_{\gamma}^{2,h}(\mathcal{O}))$ for all $\mathcal{O} \subset \Omega$. Therefore, $v^w \rightarrow v$ in $L^2(\tau, T; (H_{\gamma}^{2,h})_{\text{loc}}(\Omega))$ for a suitable v .

By STEP 1 we have that $\{v^m\}$ is also bounded in $L^2(\tau, T; H_{\gamma}^{3,h})$ (let us recall that $v^m|_{\mathcal{O}}$ is the restriction of v^m to \mathcal{O}), and hence $v^m \rightharpoonup \tilde{v}$ weakly in $L^2(\tau, T; H_{\gamma}^{3,h}(\Omega))$. Thus, due to the uniqueness of the limit, one has that $(\tilde{v})|_{\mathcal{O}} = v$ on every ball $\mathcal{O} \subset \Omega$. Therefore v is defined on Ω and $v \in L^2(\tau, T; H_{\gamma}^{3,h}(\Omega))$.

STEP 5: *The limiting function v is a weak solution.*

We now show that v is a weak solution of problem (1.1). Hence, we have to check that, for any $w_j \in H_{\gamma}^{3,h} \cap H_0^2$ (the elements of the basis for the considered test functions), the weak formulation for v^m passes to the limit as $m \rightarrow +\infty$. It is enough to verify that the nonlinear term passes to the limit, i.e., setting

$\Omega' = \text{supp } w_j \cap \Omega$, we take into account the difference

$$\begin{aligned}
& \left| \int_{\tau}^T (B(v^m, v^m), \psi^2 w_j \sigma) dt - \int_{\tau}^T (B(v, v), \psi^2 w_j \sigma) dt \right| \\
&= \left| \int_{\tau}^T (B(v^m, v^m - v), \psi^2 w_j \sigma) dt + \int_{\tau}^T (B(v^m - v, v), \psi^2 w_j \sigma) dt \right| \\
&= \left| \int_{\tau}^T \int_{\Omega} [\partial_1(v^m - v) \Delta v^m \partial_2(\psi^2 w_j) - \partial_2(v^m - v) \Delta v^m \partial_1(\psi^2 w_j)] \sigma dx dt \right. \\
&\quad \left. + \int_{\tau}^T \int_{\Omega} [\partial_1 v \Delta(v^m - v) \partial_2(\psi^2 w_j) - \partial_2 v \Delta(v^m - v) \partial_1(\psi^2 w_j)] \sigma dx dt \right| \\
&= \left| \int_{\tau}^T \int_{\Omega'} [\partial_1(v^m - v) \Delta v^m \partial_2(\psi^2 w_j) - \partial_2(v^m - v) \Delta v^m \partial_1(\psi^2 w_j)] \sigma dx dt \right. \\
&\quad \left. + \int_{\tau}^T \int_{\Omega'} [\partial_1 v \Delta(v^m - v) \partial_2(\psi^2 w_j) - \partial_2 v \Delta(v^m - v) \partial_1(\psi^2 w_j)] \sigma dx dt \right|.
\end{aligned}$$

We estimate singularly the above terms, so that

$$\begin{aligned}
& \left| \int_{\tau}^T \int_{\Omega'} [\partial_1(v^m - v) \Delta v^m \partial_2(\psi^2 w_j)] \sigma dx dt \right| \\
&\leq C \int_{\tau}^T \int |\partial_1(v^m - v)| |\psi \Delta v^m| (|\psi \partial_2 w_j| + \epsilon |\psi w_j|) dx dt \\
&\leq C \int_{\tau}^T \|\psi \Delta v^m\| \|\partial_1(v^m - v)\|_{L^4} (\|\psi \partial_2 w_j\|_{L^4} + \epsilon \|\psi w_j\|_{L^4}) dt \\
&\leq C \|w_j\|_{H_{\gamma}^{3,h}} \int_{\tau}^T \|\psi \Delta v^m\| \|\partial_1 \nabla(v^m - v)\| dt.
\end{aligned}$$

Again, for the second term

$$\begin{aligned}
& \left| \int_{\tau}^T \int_{\Omega'} \partial_2(v^m - v) \Delta v^m \partial_1(\psi^2 w_j) \sigma dx dt \right| \\
&\leq C \int_{\tau}^T \int |\partial_1 \partial_2(v^m - v)| |\psi \Delta v^m| |\psi w_j| dx dt \\
&\quad + C \int_{\tau}^T \int |\partial_2(v^m - v)| |\psi \partial_1 \Delta v^m| |\psi w_j| dx dt \\
&\leq C \|\psi w_j\|_{L^{\infty}} \left[\int_{\tau}^T \|\partial_1 \nabla(v^m - v)\| \|\psi \Delta v^m\| dt \right. \\
&\quad \left. + \int_{\tau}^T \|\nabla(v^m - v)\| \|\psi \partial_1 \Delta v^m\| dt \right]
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_{\tau}^T \int_{\Omega'} [\partial_1 v \Delta(v^m - v) \partial_2(\psi^2 w_j) \sigma] dx dt \right| \\
& \leq C \int_{\tau}^T \int |\psi \partial_1 \nabla v| |\nabla(v^m - v)| (|\psi \partial_2 w_j| + |\epsilon \psi w_j|) dx dt \\
& \quad + C \int_{\tau}^T \int |\psi \partial_1 v| |\nabla(v^m - v)| (|\epsilon \psi w_j| + |\epsilon \psi \partial_2 w_j| + |\epsilon \psi \nabla w_j| + |\psi \partial_2 \nabla w_j|) dx dt \\
& \leq C \int_{\tau}^T \|\nabla(v^m - v)\| \|\psi \partial_1 \nabla v\|_{L^4} (\|\psi \partial_2 w_j\|_{L^4} + \|\epsilon \psi w_j\|_{L^4}) dx dt \\
& \quad + C \int_{\tau}^T \|\psi \partial_1 v\|_{L^\infty} \|\nabla(v^m - v)\| (\|\epsilon \psi w_j\| + \|\epsilon \psi \partial_2 w_j\| + \|\epsilon \psi \nabla w_j\| + \|\psi \partial_2 \nabla w_j\|) dx dt \\
& \leq C \|w_j\|_{H_{\gamma}^{3,h}} \int_{\tau}^T \|\nabla(v^m - v)\| (\|\psi \partial_1 \nabla v\| + \|\psi \partial_1 \Delta v\|) dt.
\end{aligned}$$

Analogously, we have that

$$\begin{aligned}
& \left| \int_{\tau}^T \int_{\Omega'} \partial_2 v \Delta(v^m - v) \partial_1(\psi^2 w_j) \sigma dx dt \right| \\
& \leq C \int_{\tau}^T \int_{\Omega'} |\partial_2 \nabla v| |\nabla(v^m - v)| |\partial_1(\psi^2 w_j)| dx dt \\
& \quad + C \int_{\tau}^T \int_{\Omega'} |\partial_2 v| |\nabla(v^m - v)| |\partial_1 \nabla(\psi^2 w_j)| dx dt \\
& \leq C \|w_j\|_{H_{\gamma}^{3,h}} \int_{\tau}^T \|\nabla(v^m - v)\| \|\psi \partial_2 \nabla v\| dt \\
& \quad + \int_{\tau}^T \|\nabla(v^m - v)\| \|\psi \partial_2 v\|_{L^4} (\|\epsilon \psi w_j\|_{L^4} + \|\epsilon \psi \partial_1 w_j\|_{L^4} \\
& \quad + \|\epsilon \psi \nabla w_j\|_{L^4} + \|\psi \partial_1 \nabla w_j\|_{L^4}) dt \\
& \leq C \|w_j\|_{H_{\gamma}^{3,h}} \int_{\tau}^T \|\nabla(v^m - v)\| \|\psi \partial_2 \nabla v\| dt.
\end{aligned}$$

Therefore, all the above four terms go to 0 as m goes to $+\infty$.

STEP 6: *Continuity, i.e. $v \in C([0, T], H_{\gamma}^{1,h})$, $T > 0$.*

Fix any $T > 0$. First, we notice that $v_t, \partial_1 v_t \in L^2(0, T; H_{\gamma}^1(\Omega)')$ since, thanks to Theorem 1.1, they belong to $L^2(0, T; H^1(\Omega))$ and $H_{\gamma}^1 \subset H^1 \subset (H_{\gamma}^1)'$. Moreover, from Theorem 3.1, we have in particular that $v, \partial_1 v \in L^2(0, T; H_{\gamma}^1(\Omega))$. By interpolation, we conclude that $v, \partial_1 v \in C([0, T]; L_{\gamma}^2(\Omega))$, which is the claim. \square

APPENDIX A. PROPERTIES OF THE WEIGHT FUNCTION

First, we introduce the function $\phi(x, \epsilon, \gamma) = \phi(x, 3, 2, \epsilon, \gamma) = (1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{\gamma}$ and observe that

$$(A.1) \quad |\partial^{\beta} \phi| \leq C \epsilon^{|\beta|} \phi^{1/2},$$

for every multi-index $\beta = (\beta_1, \beta_2)$, $0 < |\beta| \leq 3$ and $\beta_2 \leq 2$, provided $0 < \gamma \leq 2/3$.

Actually, $|\partial_1 \phi| \leq 3\epsilon \phi^{1/2}$ if and only if

$$|\partial_1(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^\gamma| = \frac{3\gamma\epsilon|\epsilon x_1|^2}{(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{1-\gamma}} \leq 3\epsilon(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{\gamma/2},$$

which is satisfied provided that

$$(A.2) \quad 2 \leq 3 \left(1 - \frac{1}{2}\gamma\right) \iff 0 < \gamma \leq \frac{2}{3}.$$

Clearly, under this hypothesis we also have that $|\partial_2 \phi| \leq 3\epsilon \phi^{1/2}$.

Moreover, we have that the following relation holds true:

$$\begin{aligned} |\partial_1^2(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^\gamma| &\leq \frac{6\epsilon^2|\epsilon x_1|}{(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{1-\gamma}} + \frac{9(1-\gamma)\epsilon^2|\epsilon x_1|^4}{(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{2-\gamma}} \\ &\leq 15\epsilon^2(1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{\gamma/2}, \end{aligned}$$

provided that $\gamma \leq 4/3$. Under this last condition one can easily check that $|\partial^\beta \phi| \leq C\epsilon^{|\beta|}\phi^{1/2}$ for every multi-index $\beta = (\beta_1, \beta_2)$, $|\beta| \leq 2$. More in general, taking $\gamma \leq 2/3$, as in (A.2), we have (A.1).

Now, consider the map $g(\tau)$, $\tau \geq 0$ (see [6, (3.5), p. 561]), given by

$$(A.3) \quad \begin{aligned} g(\tau) &= 1/4 + \tau^2, & 0 \leq \tau \leq 1/2 \\ g(\tau) &= \tau, & 1/2 \leq \tau \leq \rho \\ g(\tau) &= \rho + 1/2 - (\rho + 1 - \tau)^2/2, & \rho \leq \tau \leq \rho + 1 \\ g(\tau) &= \rho + 1/2, & \tau \geq \rho + 1, \end{aligned}$$

and obviously $g(\tau) = \tau$ when $\rho = +\infty$, $\tau \geq 1/2$.

Define the weight function φ as

$$(A.4) \quad \varphi(x_1, x_2, \epsilon, \rho, \gamma) := \left(g((1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{1/2})\right)^{2\gamma};$$

then it holds true that

$$\lim_{\rho \rightarrow +\infty} \varphi(x_1, x_2, \epsilon, \rho, \gamma) = (1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^\gamma.$$

We are ready to show Lemma 5.1.

Sketch of the proof of Lemma 5.1. This is due, essentially, to the fact that $0 \leq g' \leq 1$, $g'(\tau) \equiv 0$ when $\tau > \rho + 1$, to $g(\tau) \sim \tau$ when $1/2 \leq \tau \leq \rho + 1$, and to the properties of the weight ϕ when $\gamma \leq 2/3$.

Take

$$\tau = \phi(x, 3, 2, \epsilon, 1/2) = (1 + |\epsilon x_1|^3 + |\epsilon x_2|^2)^{1/2}$$

and begin by considering $|\beta| = 1$, with $\partial^\beta = \partial_1$ (the case of $\partial^\beta = \partial_2$ is easier and left to the reader). We want to prove that

$$(A.5) \quad |\partial_1 \varphi| = 2\gamma g(\phi)^{2\gamma-1} g'(\phi) |\partial_1 \phi| \leq C\epsilon \varphi^{1/2},$$

where $\phi := \phi(x, 3, 2, 1/2)$.

From $|\partial_1 \phi| = 3\epsilon|\epsilon x_1|^2/2\phi \leq C\epsilon|\epsilon x_1|^{1/2}$ and the definition (A.3), we get

$$|\partial_1 \varphi| \leq \begin{cases} \gamma\epsilon C(1/4 + \phi^2)^{2\gamma-1} |\epsilon x_1|^{1/2}, & 0 \leq \phi \leq 1/2 \\ \gamma\epsilon C\phi^{2\gamma-1} |\epsilon x_1|^{1/2}, & 1/2 \leq \phi \leq \rho \\ \gamma\epsilon C\left(\rho + 1/2 - (\rho + 1 - \phi)^2/2\right)^{2\gamma-1} (\rho + 1 - \phi) |\epsilon x_1|^{1/2}, & \rho \leq \phi \leq \rho + 1 \\ 0 & \phi \geq \rho + 1. \end{cases}$$

Consider the case of $\partial_1 \varphi$ when $\rho \leq \phi \leq \rho + 1$, the others are similar.

The condition $|\partial_1 \varphi| \leq \epsilon C \varphi^{1/2}$ yields if we show that

$$\gamma \epsilon \left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^{2\gamma-1} (\rho + 1 - \phi) |\epsilon x_1|^{1/2} \leq \epsilon C \left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^\gamma,$$

i.e.

$$\gamma \left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^{\gamma-1} (\rho + 1 - \phi) |\epsilon x_1|^{1/2} \leq C.$$

Hence, recalling that $\rho \leq \phi \leq \rho + 1$, we obtain

$$(A.6) \quad \frac{(\rho + 1 - \phi) |\epsilon x_1|^{1/2}}{\left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^{1-\gamma}} \leq \frac{|\epsilon x_1|^{1/2}}{\left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^{1-\gamma}} \\ \leq \frac{|\epsilon x_1|^{1/2}}{\rho^{1-\gamma}} \leq \frac{\phi^{1/3}}{\rho^{1-\gamma}} \leq \frac{(1 + \rho)^{1/3}}{\rho^{1-\gamma}}$$

which is bounded for $\gamma \leq 2/3$. Then, relation (A.5) follows.

Now, consider the case of the second derivatives (actually we take into account just the mixed partial derivatives $\partial^\beta \varphi = \partial_{12}^2 \varphi$). We have that

$$\begin{aligned} \partial_2 \left(2\gamma g(\phi)^{2\gamma-1} g'(\phi) \partial_1 \phi \right) &= 2\gamma(2\gamma-1) g(\phi)^{2\gamma-2} (g'(\phi))^2 \partial_2 \phi \partial_1 \phi \\ &\quad + 2\gamma g(\phi)^{2\gamma-1} g''(\phi) \partial_2 \phi \partial_1 \phi + 2\gamma g(\phi)^{2\gamma-1} g'(\phi) \partial_{12}^2 \phi \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Let $\rho \leq \phi \leq \rho + 1$. In what follows we develop the calculations only for this case; the others are similar, if not more elementary.

We prove that $|A_i| \leq C \epsilon^2 \varphi^{1/2}$, for $i = 1, 2, 3$. First consider A_1 . Recalling that x_2 is bounded, we have that

$$|A_1| \leq C \epsilon^2 g(\phi)^{2\gamma-2} (\rho + 1 - \phi)^2 |\epsilon x_1|^{1/2} \leq C \epsilon^2 g(\phi)^{2\gamma-2} |\epsilon x_1|^{1/2},$$

and hence condition $|A_1| \leq C \epsilon^2 \varphi^{1/2}$ follows by requiring

$$(A.7) \quad \epsilon^2 g(\phi)^{2\gamma-2} |\epsilon x_1|^{1/2} \leq C \epsilon^2 g(\phi)^\gamma \text{ or equivalently } g(\phi)^{\gamma-2} |\epsilon x_1|^{1/2} \leq C,$$

with $\rho \leq \phi \leq \rho + 1$. Exploiting the same computations as in (A.6), we obtain

$$\frac{|\epsilon x_1|^{1/2}}{\left(\rho + 1/2 - (\rho + 1 - \phi)^2/2 \right)^{2-\gamma}} \leq \frac{|\epsilon x_1|^{1/2}}{\rho^{2-\gamma}} \leq \frac{\phi^{1/3}}{\rho^{2-\gamma}} \leq \frac{(1 + \rho)^{1/3}}{\rho^{2-\gamma}} \leq C,$$

where the last inequality is satisfied for $\gamma \leq 5/3$.

As for A_2 , one can see that $|A_2| \leq C \epsilon^2 g(\phi)^{2\gamma-1} |\epsilon x_1|^{1/2}$. So $|A_2| \leq C \epsilon^2 \varphi^{1/2}$ if

$$\epsilon^2 g(\phi)^{2\gamma-1} |\epsilon x_1|^{1/2} \leq C \epsilon^2 g(\phi)^\gamma \text{ or equivalently } g(\phi)^{\gamma-1} |\epsilon x_1|^{1/2} \leq C,$$

which is certainly verified under the assumption that $\gamma \leq 2/3$.

Since $|A_3| \leq C g(\phi)^{2\gamma-1} |\partial_{12}^2 \phi|$, the condition $|A_3| \leq C \epsilon^2 \varphi^{1/2}$ is satisfied provided that

$$(A.8) \quad \epsilon^2 g(\phi)^{2\gamma-1} \frac{|\epsilon x_1|^2 |\epsilon x_2|}{\phi^3} \leq C \epsilon^2 g(\phi)^\gamma, \text{ that is } g(\phi)^{\gamma-1} \frac{|\epsilon x_1|^2 |\epsilon x_2|}{\phi^3} \leq C.$$

We conclude using the fact that $\gamma \leq 2/3$ and that

$$\frac{|\epsilon x_1|^2 |\epsilon x_2|}{\phi^3} \leq \frac{\phi^{7/3}}{\phi^3} \leq 1.$$

Using the previous computations along with the fact that $g'''(\tau) \equiv 0$, the general case (5.1) follows directly. \square

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